Reference-based multiple criteria sorting-ranking

V. Mousseau[1], M. Pirlot[2], and O. Sobrie[2]

[1] MICS, CentraleSupélec, Université Paris-Saclay, France
[2] MATHRO, Faculté Polytechnique, UMONS, Belgium

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Motivation

Absolute vs comparative assessment

Sorting is frequently relevant...

... but sometimes insufficient

Necessity to discriminate further than the partition into classes

- bi-partition + rank the top class
- bi-partition + rank the bottom class

Motivation

Our aim : S & R

Class of methods to **Sort&Rank** alternatives w.r.t. reference alternatives

$\geq_i$ a complete preorder on X_i, $i \in \mathcal{N} = \{1, \dots, n\}$

$X = \prod_{i \in \mathcal{N}} X_i$

dominance relation $x \geq y$ if $x_i \geq_i y_i$, $i \in \mathcal{N}$

Primitives

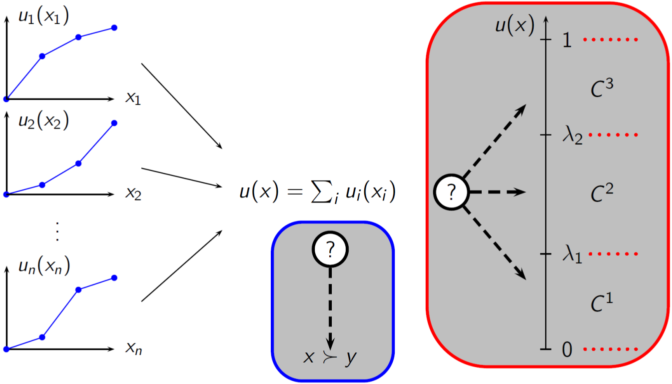
- $C^1, \dots, C^p$ an ordered partition of X,
  $cat(x) \in \mathcal{P}$, $x \in X$, where $\mathcal{P} = \{1, \dots, p\}$
  $x \geq y \Rightarrow cat(x) \geq cat(y)$
- ranking $\succsim$ on X, a complete preorder
  $x \geq y \Rightarrow x \succsim y$
- cat-$\succsim$-compatibility : $cat(x) > cat(y) \Rightarrow x \succ y$

A FIRST SIMPLE S & R EXAMPLE

Additive value-based S & R

$$cat(x) = h \quad \text{if} \quad \lambda_{h-1} \leq \sum_i u_i(x_i) < \lambda_h$$

$$x \succsim y \quad \text{if} \quad \sum_i u_i(x_i) \geq \sum_i u_i(y_i)$$



Our aim

A reference-based non-compensatory S & R model

- Sort alternatives in ordered categories using an NCS model
- Rank alternatives assigned to each category using an RMP model

Contents

THE NONCOMPENSATORY SORTING MODEL (NCS)

- MCDA method based on outranking relations
- Characterized by [Bouyssou and Marchant, 2007]

An object is assigned to a category if :

- It is better than the lower limit of the category on a sufficiently strong subset of criteria
- While this is not the case when comparing the object to the upper limit of the category

Noncompensatory Sorting (2 categories)

Simplest case : 2 categories

- 2 categories : Good ($\mathcal{G}$), Bad ($\mathcal{B}$)
- alternatives to be sorted : $X = \prod_{i \in \mathcal{N}} X_i$ with $\mathcal{N} = \{1, \ldots, n\}$
- $\geq_i$ total preorder on X_i
- limit profile $b = (b_1, \ldots, b_n)$
- $\mathcal{F}$ = family of sufficient coalitions, which is a subset of $\underline{2}^{\mathcal{N}}$ up-closed by inclusion

Assignment rule :

For all $x = (x_1, \ldots, x_n) \in X, \quad x \in \mathcal{G} \quad \text{iff} \quad \{i \in \mathcal{N} : x_i \geq_i b_i\} \in \mathcal{F}$

Noncompensatory Sorting (More than 2 categories)

- an ordered set $C^1 \prec \cdots \prec C^p$ of p categories.
- alternatives to be sorted : $X = \prod_{i \in \mathcal{N}} X_i$ with $\mathcal{N} = \{1, \ldots, n\}$
- $\geq_i$ total preorder on X_i, $i \in \mathcal{N}$,
- limit profiles $b^h = (b_1^h, \ldots, b_n^h)$ such that $b_i^h \geq_i b_i^{h-1}$, with $i \in \mathcal{N}, h = 1..p-1$
- b^h is the upper limit of C^h, and the lower limit of C^{h+1}
- $p-1$ embedded families of sufficient coalitions $\mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq ... \supseteq \mathcal{F}^{p-1}$ (subsets of $\underline{2}^{\mathcal{N}}$ up-closed by inclusion).

Assignment rule :

For all $x = (x_1, \ldots, x_n) \in X$

$$x \in C^h \quad \text{iff} \quad \{i \in \mathcal{N} : x_i \geq_i b_i^{h-1}\} \in \mathcal{F}^{h-1} \text{ and } \{i \in \mathcal{N} : x_i \geq_i b_i^h\} \notin \mathcal{F}^h$$

in the following, we denote $c(x, b^h) = \{i \in \mathcal{N} : x_i \geq_i b_i^h\}$

Contents

Ranking with Multiple Profiles (RMP)

[Rolland 2008,2013],

[Bouyssou, Marchant 2013].

RMP (with two profiles) :

x is preferred to y if

- The coalition of criteria for which x is over the lower profile is stronger than the one on which y is over the lower profile,
- in case of equality, the same comparison is performed w.r.t. the upper profile.

One can perform the comparisons in the reverse order.

Ranking with Multiple Profiles (RMP)

With two profiles

- alternatives to be ranked : $X = \prod_{i \in \mathcal{N}} X_i$ with $\mathcal{N} = \{1, \ldots, n\}$
- $\geq_i$ total preorder on X_i, $i \in \mathcal{N}$,
- 2 limit profiles $b = (b_1, \ldots, b_n)$ and $b^{'} = (b_1^{'}, \ldots, b_n^{'})$ such that $b^{'} \geq b$,
  an importance relation $\trianglerighteq \subset 2^{\mathcal{N}} \times 2^{\mathcal{N}}$.
- $\trianglerighteq$ is monotone w.r.t. inclusion, i.e, $A \trianglerighteq B$, $A \subseteq C$, and $D \subseteq B$ then $C \trianglerighteq D$,

RMP (with two profiles) :

For all $x, y \in X$

$$x \succsim y \quad \text{iff} \quad c(x,b) \trianglerighteq c(y,b) \ \text{ or } \ [c(x,b) \triangleq c(y,b) \text{ and } c(x,b^{'}) \trianglerighteq c(y,b^{'})]$$

with two profiles, two RMP variants exist :

compare to b then to $b^{'}$.

compare to $b^{'}$ then to b .

Ranking with Multiple Profiles (RMP)

With ℓ profiles

alternatives to be ranked : $X = \prod_{i \in \mathcal{N}} X_i$ with $\mathcal{N} = \{1, \ldots, n\}$

- $\geq_i$ total preorder on X_i, $i \in \mathcal{N}$,
- ℓ limit profiles $b^h = (b^h_1, \ldots, b^h_n)$ for $h \in L = \{1, \ldots, \ell\}$, such that $b^{h+1} \geq b^h$, for $h = 1, \ldots, \ell - 1$
- an importance relation $\trianglerighteq_L$ **on ℓ-tuples** of subsets of $\mathcal{N}$
- $\trianglerighteq_L$ is monotone w.r.t. inclusion on each dimension

RMP (with ℓ *profiles*) :

For all $x, y \in X$, $\quad x \succsim y \quad$ iff

$$\big(c(x, b^1), \ldots, c(x, b^h), \ldots, c(x, b^\ell)\big) \trianglerighteq_L \big(c(y, b^1), \ldots, c(y, b^h), \ldots, c(y, b^\ell)\big)$$

$\trianglerighteq_L$ compares how alternatives are positioned w.r.t. each reference point

Example of implementation of $\trianglerighteq_L$:

define relations $\trianglerighteq_h \subseteq 2^{\mathcal{N}} \times 2^{\mathcal{N}}$ that compare coalitions w.r.t. to each reference b^h

use them lexicographically considering the profiles in a certain order

Contents

Reference based S & R model

S & R : principles

Sort alternatives in categories using an NCS model,

using limit profiles $b^h,\ h = 1, \ldots p-1$,

and families of sufficient coalitions $\mathcal{F}^h$ with respect to profile b^h.

Rank alternatives by means of an RMP model

using the limit profiles $b^h,\ h = 1, \ldots p-1$ as the reference points

and a relation $\trianglerighteq_L$ defined in a particular way that takes into account the families of sufficient coalitions $\mathcal{F}^h$

Reference based S & R model

Primitives

- category assignment $cat(x) \in \mathcal{P}$, where $\mathcal{P} = \{1, ..., p\}$
- ranking $\succsim$ on X, a complete preorder
- cat-$\succsim$-compatibility : $cat(x) > cat(y) \Rightarrow x \succ y$

NCS parameters

For assigning the objects to categories :

- profiles : $b^1 < b^2 < \ldots < b^{p-1}$;
- sufficient coalitions : $\mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq ... \supseteq \mathcal{F}^{p-1}$ (subsets of $\underline{2}^{\mathcal{N}}$ up-closed by inclusion).

RMP parameters

For ranking the objects :

- reference points = profiles $b^1 \ldots, b^h, \ldots b^{p-1}$;
- relations $\trianglerighteq_h$ comparing criteria coalitions at level h (monotone w.r.t. coalitions inclusion)

Reference based S & R model

Sorting rule

$$cat(x) = h \text{ if } c(x, b^{h-1}) \in \mathcal{F}^h \text{ and } c(x, b^h) \notin \mathcal{F}^{h+1}$$

Ranking rule

(1) $cat(x) > cat(y) \Rightarrow x \succ y$;

(2) if $cat(x) = cat(y)$ then

$$x \succ y \text{ iff } [c(x, b^{h-1}) \rhd_h c(y, b^{h-1})]$$
$$\text{or } [c(x, b^{h-1}) \triangleq_h c(y, b^{h-1}) \text{ and } c(x, b^h) \rhd_{h+1} c(y, b^h)].$$

Reference based S & R model

Ranking $\succsim$: a RMP model

$\forall x, y \in X, \quad x \succsim y \quad$ if

$$\big(c(x,b^1),\dots,c(x,b^h),\dots,c(x,b^{p-1})\big) \trianglerighteq_L \big(c(y,b^1),\dots,c(y,b^h),\dots,c(y,b^{p-1})\big)$$

To define $\trianglerighteq_L$, we use

- the families of sufficient coalitions $\mathcal{F}^h$
- complete order relations $\trianglerighteq_h$ comparing coalitions at each level h (relations decreasing with h)

For ranking x and y, relation $\trianglerighteq_L$ does the following

- searches for the largest index h (resp. h') for which coalition $c(x, b^{h-1})$ (resp. $c(y, b^{h'-1})$) is sufficient, i.e. belongs to $\mathcal{F}^h$ (resp. $\mathcal{F}^{h'}$)
- if $h > h'$, then $x \succ y$ and if $h < h'$, then $y \succ x$
- if $h = h'$, then $x \succsim y$ iff $c(x, b^{h-1}) \trianglerighteq_h c(y, b^{h-1})$ or $[c(x, b^{h-1}) \triangleq_h c(y, b^{h-1})$ and $c(x, b^h) \triangleright_{h+1} c(y, b^h)]$

Reference based S & R model

NCS/RMP Compatibility

Ensure consistency between $\mathcal{F}^h$ and $\trianglerighteq_h$

- $\mathcal{F}^h$ is the set of sufficient coalitions of criteria
- $\mathcal{G}_h = 2^{\mathcal{N}} \setminus \mathcal{F}_h$ is the set of insufficient coalitions of criteria
- $\mathcal{F}^h \times \mathcal{G}_h \subset \trianglerighteq_h$
- as $\mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq \ldots \supseteq \mathcal{F}^{p-1}$, it holds $\mathcal{G}^1 \subseteq \mathcal{G}^2 \subseteq \ldots \subseteq \mathcal{G}^{p-1}$

S & R : particular cases

$\trianglerighteq_h = \trianglerighteq, \ \forall h.$

$\mathcal{F}^h = \mathcal{F}, \ \forall h.$

$\trianglerighteq_h$ representable by additive weights.

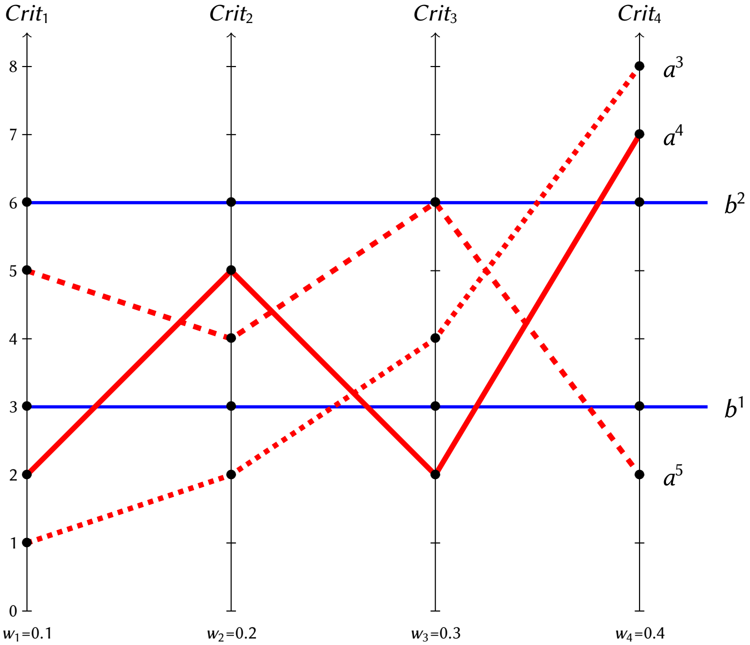
$\mathcal{F}^h$ representable by the same weights and a threshold.

Illustrative example

	$Crit_1$	$Crit_2$	$Crit_3$	$Crit_4$	
a^1	4	8	6	7	$\rightarrow C^3$
a^2	7	7	7	4	$\rightarrow C^3$
a^3	1	2	4	8	$\rightarrow C^2$
a^4	2	5	2	7	$\rightarrow C^2$
a^5	5	4	6	2	$\rightarrow C^2$
a^6	1	8	4	2	$\rightarrow C^1$
b^1	3	3	3	3	
b^2	6	6	6	6	
w_i	0.1	0.2	0.3	0.4	$\lambda = 0.6$

in C^3 it holds, $a^1 \succsim a^2$, and

in C^2 it holds, $a^3 \succsim a^4 \succsim a^5$,


$Crit_1$
$Crit_2$
$Crit_3$
$Crit_4$
8
7
6
5
4
3
2
1
0
a^3
a^4
b^2
b^1
a^5
w_1=0.1
w_2=0.2
w_3=0.3
w_4=0.4

Contents

S & R : Learning from preference

S & R : Preference data

Suppose the DM can to provide preference statements on S & R primitives

- x should be assigned to C_h ($cat(x) = h$),
- x is preferred to y ($x \succsim y$).

Learning sets

Learning sets take the following form :

- a set $A^* \subset X$ of assignment examples :
  $A^* = A_1^* \cap ... \cap A_p^*$ a partition of A^* such that $x \in A_h^*$ if $cat(a) = h$,
- a set $BC \subset X \times X$ of binary comparisons :
  $(x, y) \in BC$ when the DM prefers x to y ($x \succ y$).

S & R : Learning from preference

S & R : Learning algorithm

Input : Learning data

- a set A^* of assignment examples, and
- a set BC of binary comparisons.

Output : S & R parameters

- profiles b_1, and b_2,
- sufficient coalitions $\mathcal{F} = \{A \in 2^{\mathcal{N}} : \sum_{i \in A} w_i \geq \lambda\}$,
- $A \trianglerighteq_h B$ if $\sum_{i \in A} w_i^h \geq \sum_{i \in B} w_i^h$, $h = 1, 2, 3$,

Existing literature

- MIP to learn MR-Sort [Leroy et al., 2011], and
- MIP to learn RMP [Olteanu et al., 2019].

S & R Learning algorithm : representing assignment examples

$$\delta_j(a, b^h) = \begin{cases} 1 & \text{, if } a_j \geqslant b_j^h \\ 0 & \text{, otherwise.} \end{cases} \qquad \begin{cases} \delta_j(a, b^h) \geqslant a_j - b_j^h + \varepsilon \\ a_j - b_j^h \geqslant \delta_j(a, b^h) - 1 \end{cases}$$

$$\omega_j(a, b^h) = \begin{cases} w_j & \text{, if } a_j \geqslant b_j^h \\ 0 & \text{, otherwise.} \end{cases} \qquad \begin{cases} \omega_j(a, b^h) \geqslant 0 \\ \delta_j(a, b^h) \geqslant \omega_j(a, b^h) \\ w_j \geqslant \omega_j(a, b^h) \\ \omega_j(a, b^h) \geqslant \delta_j(a, b^h) + w_j - 1 \end{cases}$$

$$y_{a,h} = \begin{cases} 1 & \text{, if } cat(a) = h \\ 0 & \text{, otherwise.} \end{cases} \qquad \begin{cases} \sum_{j \in \mathcal{N}} \omega_j(a, b^h) - \lambda \leqslant M(1 - y_{a,h}) - \varepsilon \\ -\sum_{j \in \mathcal{N}} \omega_j(a, b^{h-1}) + \lambda \leqslant M(1 - y_{a,h}) \\ \sum_h y_{a,h} = 1, \ \forall a \end{cases}$$

S & R Learning algorithm : representing comparisons

$a \succsim a'$ if $\big[cat(a) > cat(a')$ or $[cat(a) = cat(a')$ and $a \succsim_h a']\big]$

we pose $\sum_h h.y_{a',h} \leq \sum_h h.y_{a,h}$ (a' is not in a higher category than a)

we define $\varepsilon_{a,a',h}$ equal to 1 if $cat(a) = cat(a') = h$, 0 otherwise
i.e., $\varepsilon_{a,a',h} \in \{0, 1\}$, $\varepsilon_{a,a',h} \leq y_{a,h}$, and $\varepsilon_{a,a',h} \leq y_{a',h}$, and
$\varepsilon_{a,a',h} \geq y_{a,h} + y_{a',h} - 1$.

Let us define the following expression :
$\delta_h(a, a') = \sum_{j \in \mathcal{N}} \omega_j^h(a, b^h) - \sum_{j \in \mathcal{N}} \omega_j^h(a', b^h))$

We impose that if $cat(a) = cat(a') = h$, then $\delta_{h-1}(a, a') \geq 0$
i.e, $M(\varepsilon_{a,a',h} - 1) \leq \delta_{h-1}(a, a')$

We also impose that if $cat(a) = cat(a') = h$ and $\delta_{h-1}(a, a') = 0$ then
$\delta_h(a, a') \geq 0$
i.e, $M(\varepsilon_{a,a',h} - 1 - 0.5\delta_{h-1}(a, a')) \leq \delta_h(a, a')$

Contents

S-R : Illustrative example (continued)

Illustrative example (continued)

	$Crit_1$	$Crit_2$	$Crit_3$	$Crit_4$
a^1	4	8	6	7
a^2	7	7	7	4
a^3	1	2	4	8
a^4	2	5	2	7
a^5	5	4	6	2
a^6	1	8	4	2

$a_1, a_2 \rightarrow C^3$

$a_3, a_4, a_5 \rightarrow C^2$

$a_6 \rightarrow C^1$

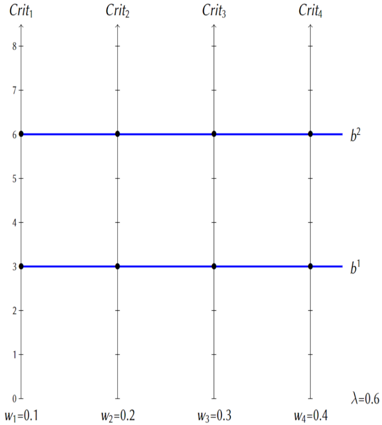
$a_1 \succ a_2$

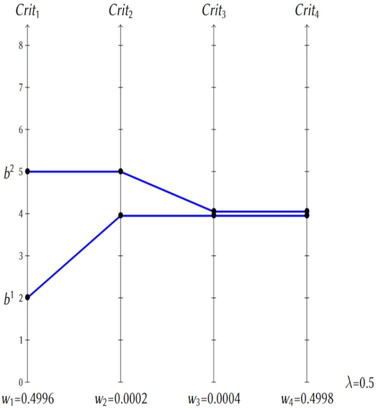
$a_3 \succ a_4 \succ a_5$

$a_1 \succ a_5, \; a_2 \succ a_3,$

$a_2 \succ a_4, \; a_4 \succ a_6$

S-R : Illustrative example (continued)





S-R : Numerical experiments

Experimental setting

We vary :

- the problem size : 3 cat./5 crit. - 4 cat./7 crit. - 5 cat./9 crit.
- the learning set size : 100, 200, 300, 400 preference statements
- the noise level : 0%, 10%

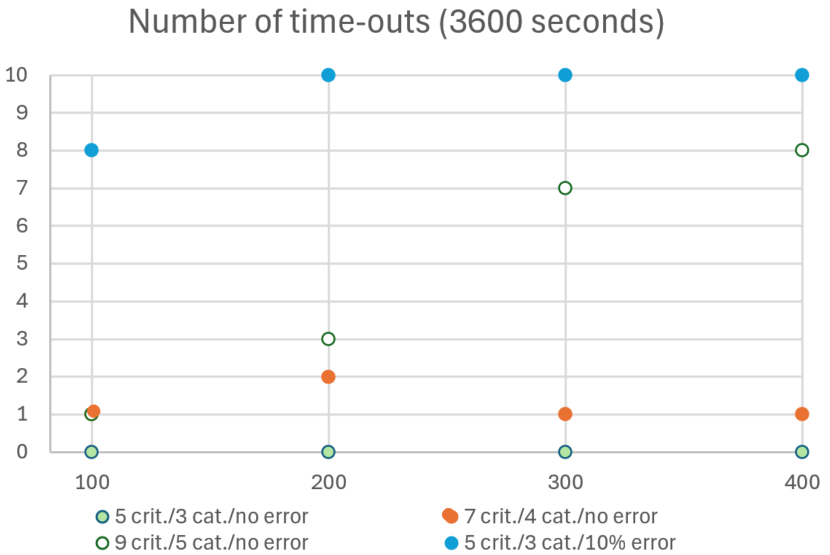
We perform 10 repetitions of each combination of parameters, using GUROBI solver with a 3600 sec. timeout

Computations

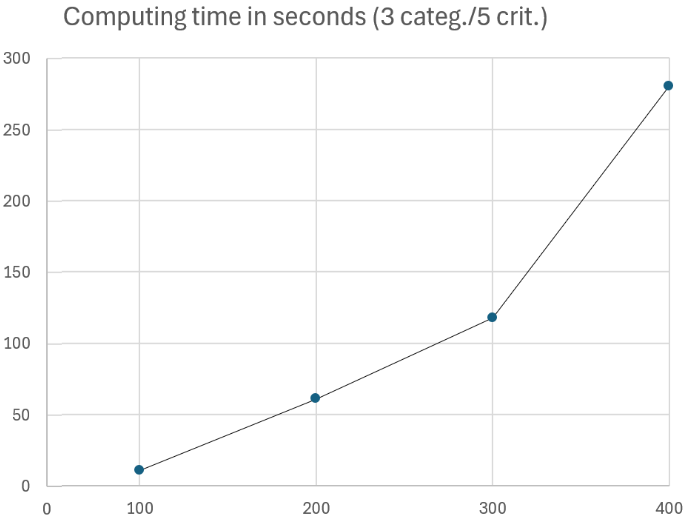
We perform 10 repetitions of each combination of parameters, using GUROBI solver with a 3600 sec. timeout and obtain :

- computing time
- classification accuracy
- pairwise comparison accuracy

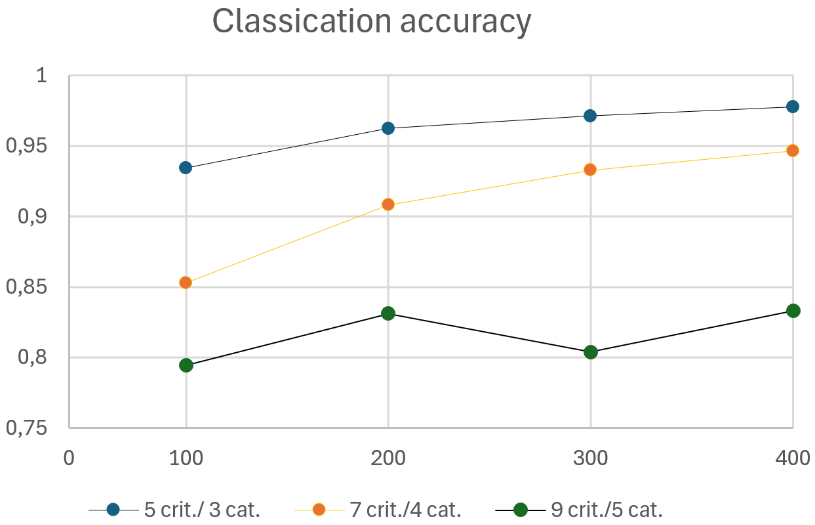
S-R : Numerical experiments



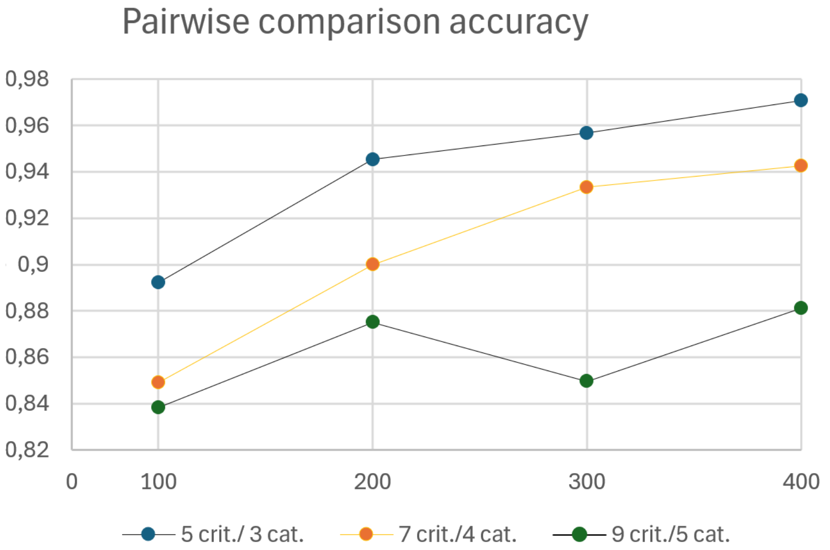
S-R : Numerical experiments



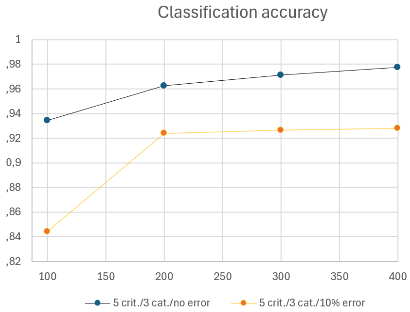
S-R : Numerical experiments

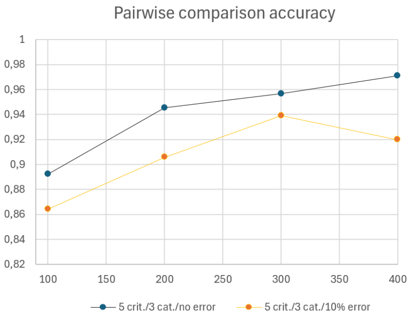


S-R : Numerical experiments



S-R : Numerical experiments





Contents

Conclusion

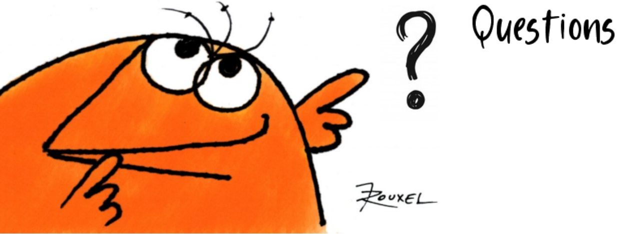
contributions

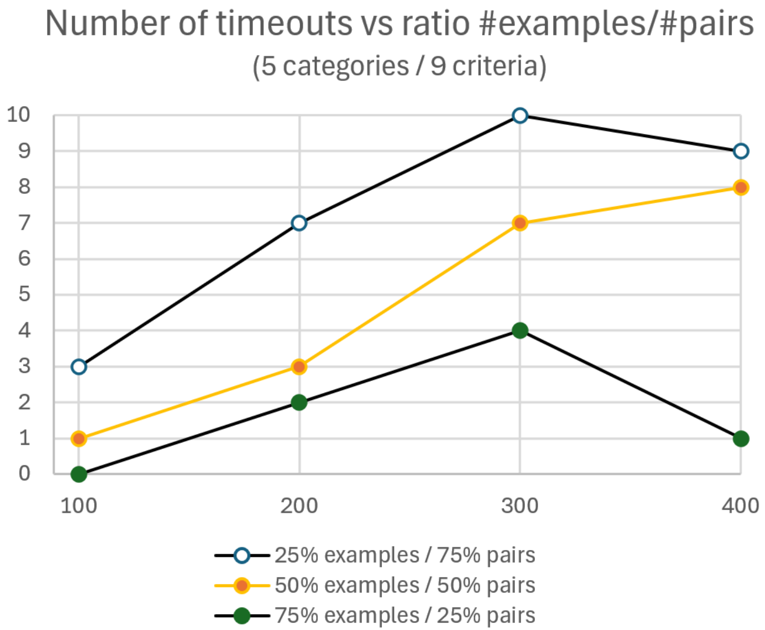
- We consider an original Sorting/Ranking problem statement,
- We propose S & R a Sorting/Ranking method using reference profiles,
- The S & R model can be learned based on preference data using a MIP-based inference algorithm.

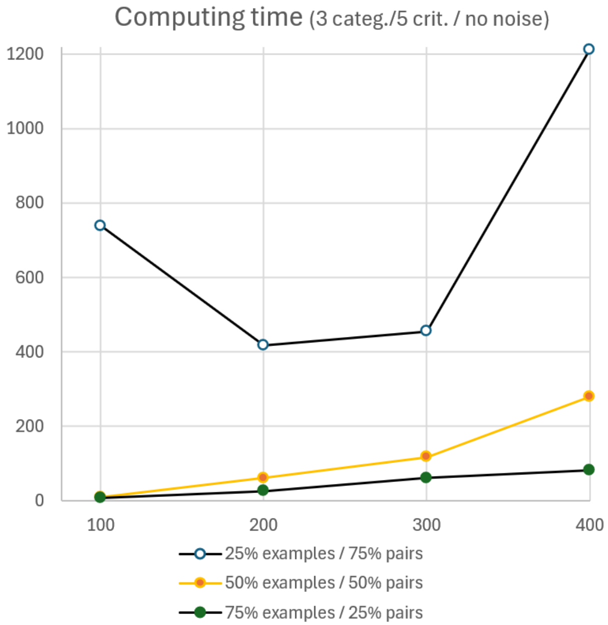
further research

- Axiomatic analysis of the S & R method,
- Extended experiments to test the behavior of the inference algorithm,
- Real-world case studies.


C'EST TOUT POUR AUJOUR D'HUI


?
Questions
ROUXEL


Number of timeouts vs ratio #examples/#pairs
(5 categories / 9 criteria)
10
9
8
7
6
5
4
3
2
1
0
100
200
300
400
25% examples / 75% pairs
50% examples / 50% pairs
75% examples / 25% pairs


Computing time (3 categ./5 crit. / no noise)
1200
1000
800
600
400
200
0
100
200
300
400
25% examples / 75% pairs
50% examples / 50% pairs
75% examples / 25% pairs