Recent developments in sensitivity analysis for PROMETHEE methods

And a novel approach for ranking analysis

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The PROMETHEE II Method

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- $A = \{a_1, ..., a_n\}$ alternatives, $G = \{g_k : A \to \mathbb{R} |\ k = 1, ..., q\}$ criteria ;

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- q preference functions $P_k : \mathcal{D}_k \to [0,1]$, *e.g.* V-shape of parameter p

$$P_k(d;p) = \begin{cases} 0 & \text{if } d \in]-\infty, 0] \\ {}^{d}/_{p} & \text{if } d \in]0, p[\\ 1 & \text{if } d \in [p, \infty[\end{cases} \quad (1)$$

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$$P_k(d;p) = \begin{cases} 0 & \text{if } d \in]-\infty, 0] \\ d/p & \text{if } d \in]0, p[\\ 1 & \text{if } d \in [p, \infty[\end{cases} \tag{1}$$

- Pairwise preferences

$$\pi(a,b) = \sum_{k=1}^{q} w_k P_k(d_k(a,b)) \tag{2}$$

Notations (cont.)

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- Net flow score (NFS):

$$\phi(a) = \phi^+(a) - \phi^-(a) \tag{5}$$

- P2 preference relation: $a \succsim b \iff \phi(a) \geq \phi(b)$

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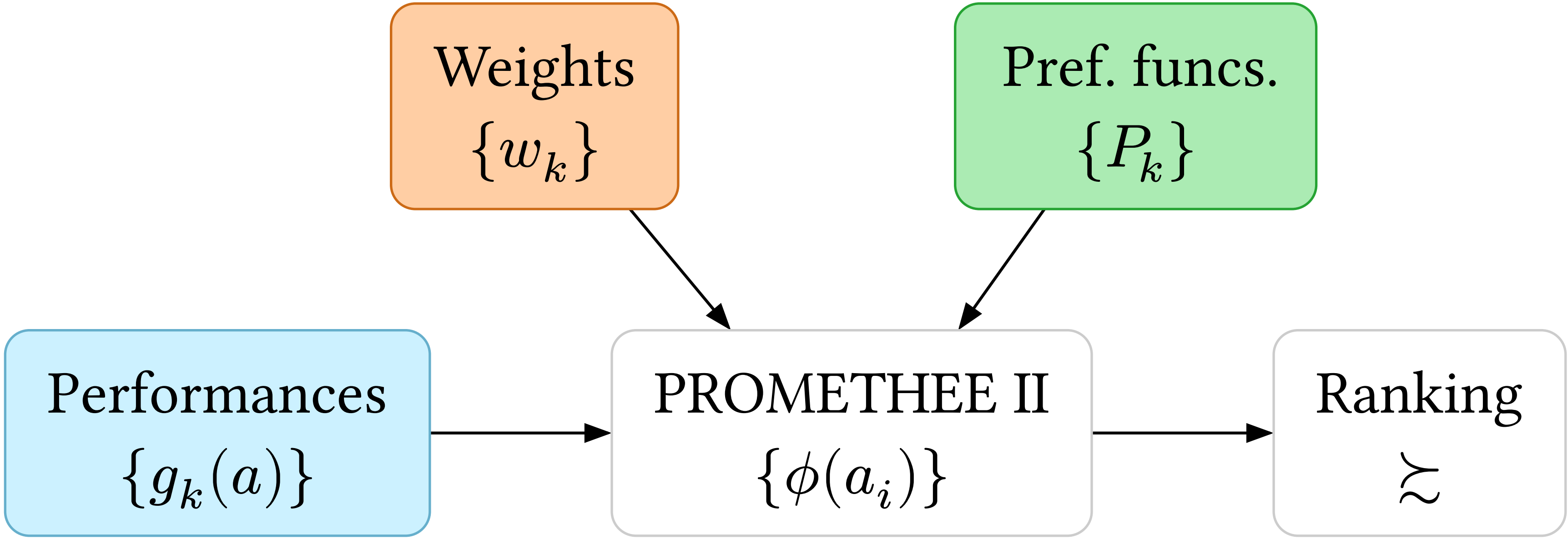
Given a table of performances and $(w_k, P_k)_{k=1}^{q}$, the P2 ranking is fixed.

Sensitivity analysis for PROMETHEE II

What and why ?

Figure 1: Representation of the ranking construction process.



Parameters are subjective and not easily defined !

Variational SA

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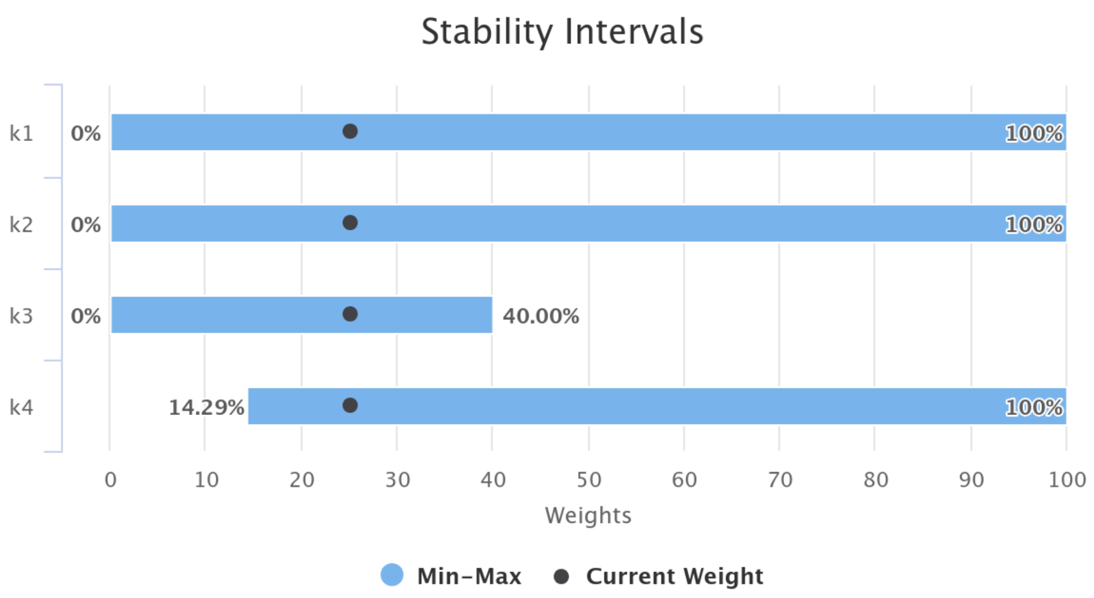
Type 2 problem Given A, G, $(w_k, P_k)_{k=1}^q$ and the corresponding P2 ranking, how much *some subset of parameters* **should change** to reach some property of the ranking ?

SA on one weight

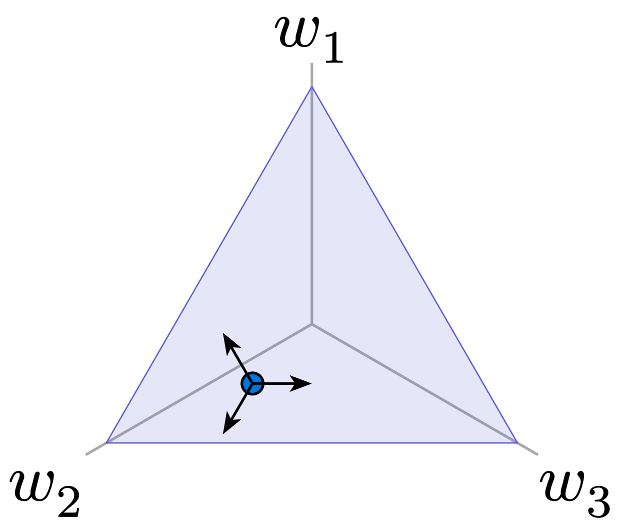
- Type 1 on **one given weight** or sum of weights was solved by B. Mareschal[1]: Weight Stability Intervals (WSI).



[1]Bertrand Mareschal, "Weight Stability Intervals in Multicriteria Decision Aid," *European Journal of Operational Research* 33, no. 1 (1988): 54–64.

SA on weights

- Type 2 problem on weights by W.T.M. Wolters and B. Mareschal[1]



- Extended by N.A.V. Doan and Y. De Smet[2]
  - Constraints on number of weights change ;
  - Consensus building.

[1] W.T.M. Wolters and Bertrand Mareschal, "Novel Types of Sensitivity Analysis for Additive MCDM Methods," *European Journal of Operational Research* 81, no. 2 (1995): 281–90.

[2] Nguyen Anh Vu Doan and Yves De Smet, "An Alternative Weight Sensitivity Analysis for PROMETHEE II Rankings," *Omega* 80 (2018): 166–74.

SA on weights

Mixed Integer Linear Programming (MILP), with $w'_k = w_k + d_k^+ - d_k^-$

$$\min\ z = \sum_{k=1}^{q} |w_k - w'_k| = \sum_{k=1}^{q} d_k^+ + d_k^- \ \text{s.t.} \tag{12}$$

$$\sum_{k=1}^{q} [\phi_k(a_i) - \phi_k(a_j)](w_k + d_k^+ - d_k^-) \geq 0 \ (\text{imposed ranking})$$

$$\sum_{k=1}^{q} (d_k^+ - d_k^-) = 0 \ (\text{normalized weights}) \tag{13}$$

$$w_k + d_k^+ - d_k^- \geq 0 \ (\text{positive weights})$$

SA on weights

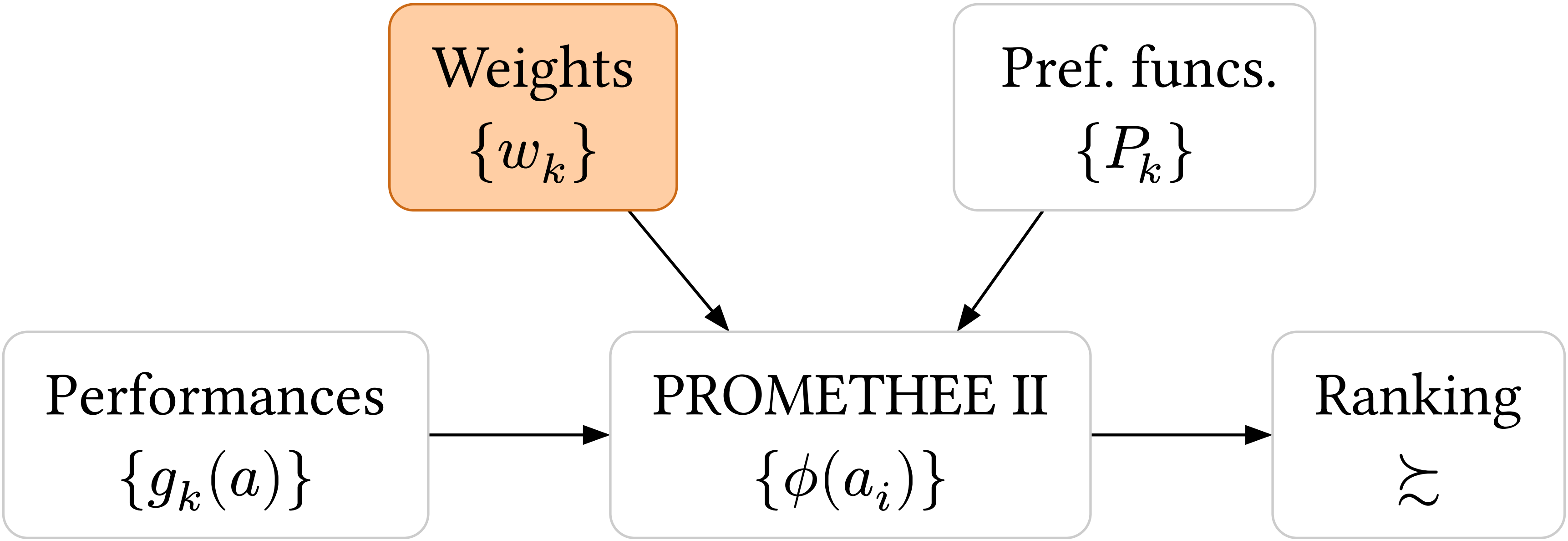


Figure 3: SA on weights.

SA on thresholds

X. Liu and Y. Liu[1]: algorithmic approach for type 1 on preference thresholds, for **one criterion** at a time.

$$P_k(d; p) = \begin{cases} 0 & \text{if } d \leq 0 \\ {}^{d}/_{p} & \text{if } 0 < d \leq p \\ 1 & \text{if } p < d \end{cases} \tag{14}$$

The relation between the ranking and thresholds is not linear (or even convex).

[1] Xianliang Liu and Yunfei Liu, "Sensitivity Analysis of the Parameters for Preference Functions and Rank Reversal Analysis in the PROMETHEE II Method," *Omega*, 2024, 103116.

SA on thresholds

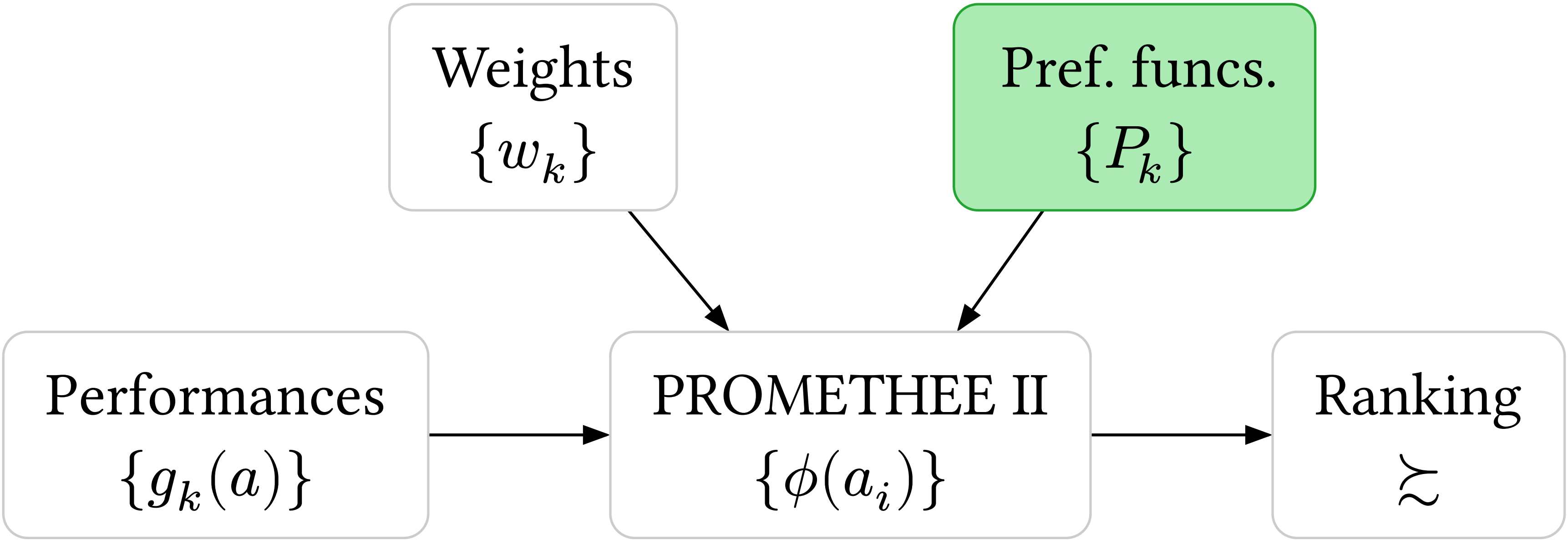


Figure 4: SA on preference thresholds.

SA on performances

- Primal and dual problems on performances mentioned by W.T.M. Wolters and B. Mareschal[1].
- A. Flachs and Y. De Smet[2]: Exact algorithmic solution for primal and dual problem for given PF shapes.

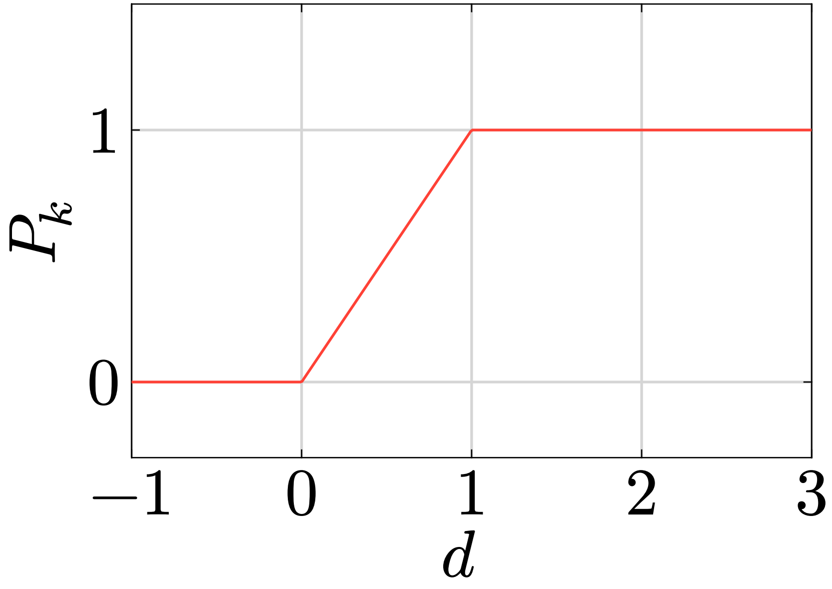


Figure 5: V-Shape preference function

[1]Wolters and Mareschal, "Novel Types of Sensitivity Analysis for Additive MCDM Methods".

[2]Alexandre Flachs and Yves De Smet, "Inverse Optimization on the Evaluations of Alternatives in the Promethee II Ranking Method," *Omega* 136 (2025): 103325.

SA on performances

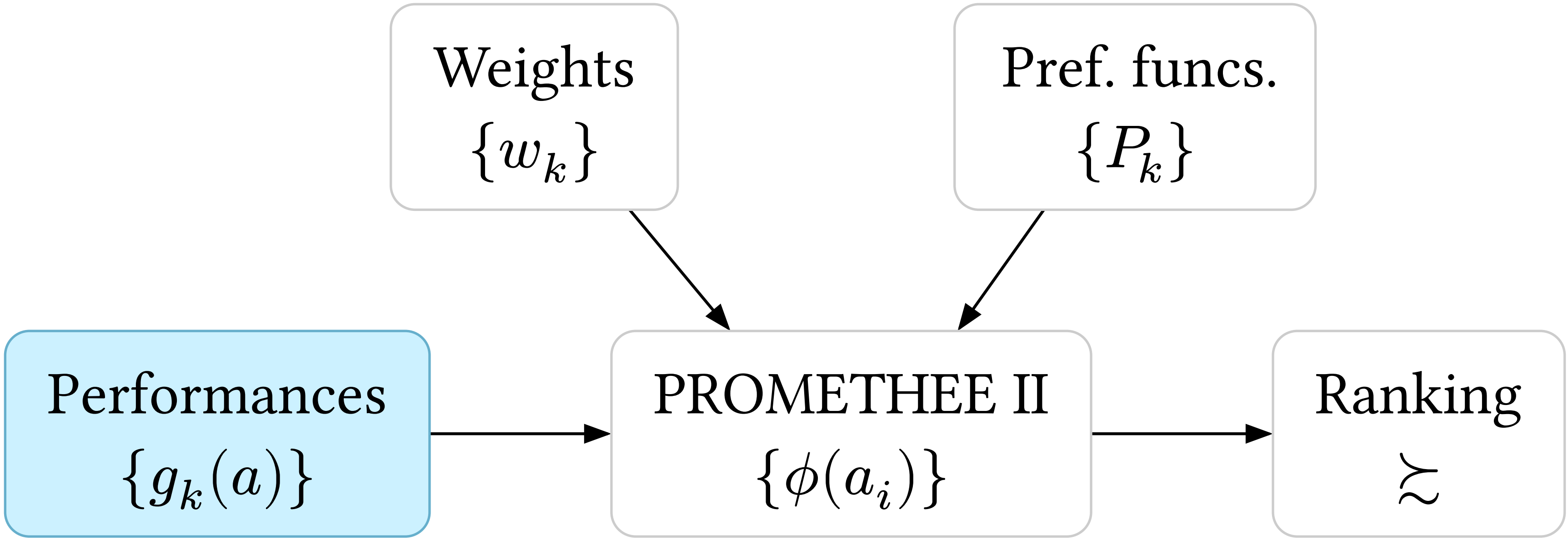


Figure 6: SA on performances.

Robust Ordinal Regression

ROR[1]: do not focus on a specific set of parameters, consider everything compatible with DM's information and deduce properties.

[1] Salvatore Greco et al., "Robust Ordinal Regression," in *Trends in Multiple Criteria Decision Analysis*, ed. Matthias Ehrgott et al. (Springer US, 2010).

[2] Miłosz Kadziński et al., "Extreme Ranking Analysis in Robust Ordinal Regression," *Omega* 40, no. 4 (2012): 488–501.

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- Necessary properties **must** be verified, *e.g.*: $a_1 \succ a_3 \Rightarrow a_6 \succ a_4$
- possible properties **could** be verified. *e.g.*:

$$\text{if } a_1 \succ a_3 \text{ it is possible that } a_5 \succ a_3 \tag{21}$$

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- This somehow encapsulates SA by nature.

Adapted for P2 by M. Kadziński, S. Greco and R. Słowiński[2].

[1] Greco et al., "Robust Ordinal Regression".

[2] Kadziński et al., "Extreme Ranking Analysis in Robust Ordinal Regression".

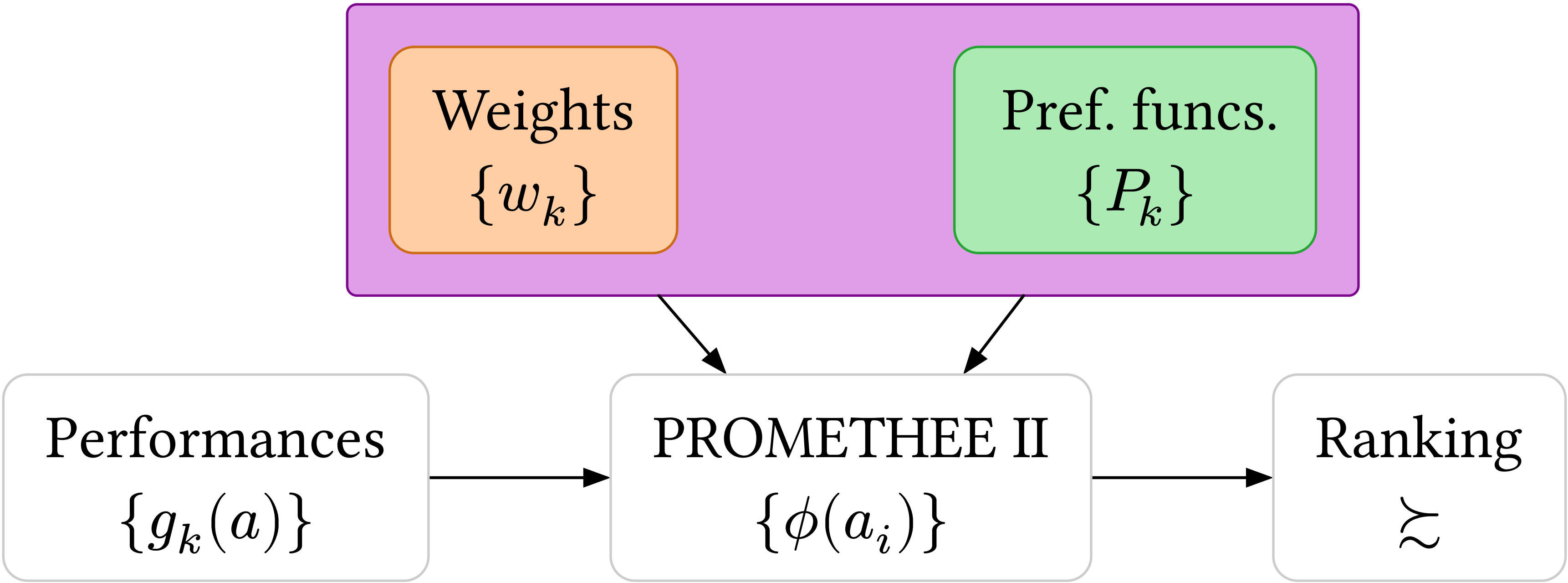

Figure 7: SA with ROR.

A Linear form of the NFS

Key observations

Order and re-label the elements in $\{|d| \mid d \in \mathcal{D}_k\}$:

$$0 = m_{k,0} < m_{k,1} < m_{k,2} < \cdots < m_{k,T_k}, \quad T_k \in \mathbb{N}. \tag{22}$$

For $k \in \{0, ..., q\}$:

$$\Delta\theta_{k,t} := P_k(m_{k,t}) - P_k(m_{k,t-1}), \quad t \in \{1, ..., T_k\} \tag{23}$$

and define

$$\mathcal{P}_k(d) := \sum_{t : m_{k,t} \leq d} \Delta\theta_{k,t} \tag{24}$$

We call $\mathcal{P}_k$ the layer-cake decomposition of P_k.

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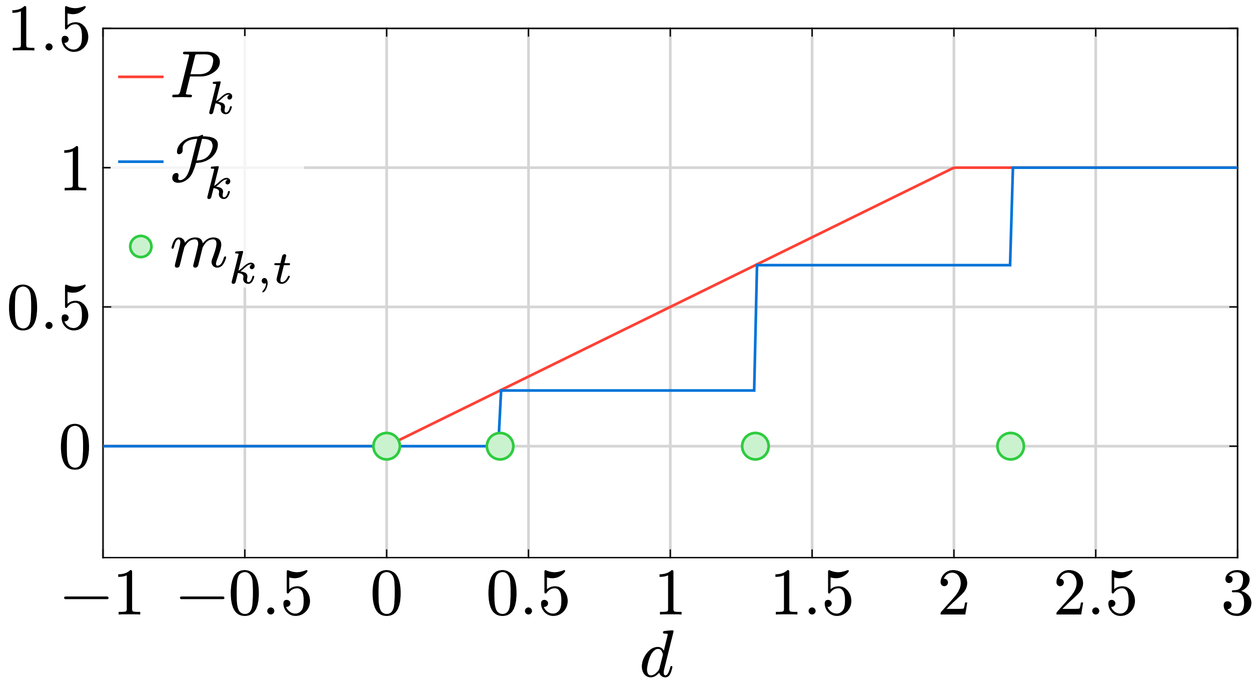

Figure 8: Layer cake preference function and preference increments.

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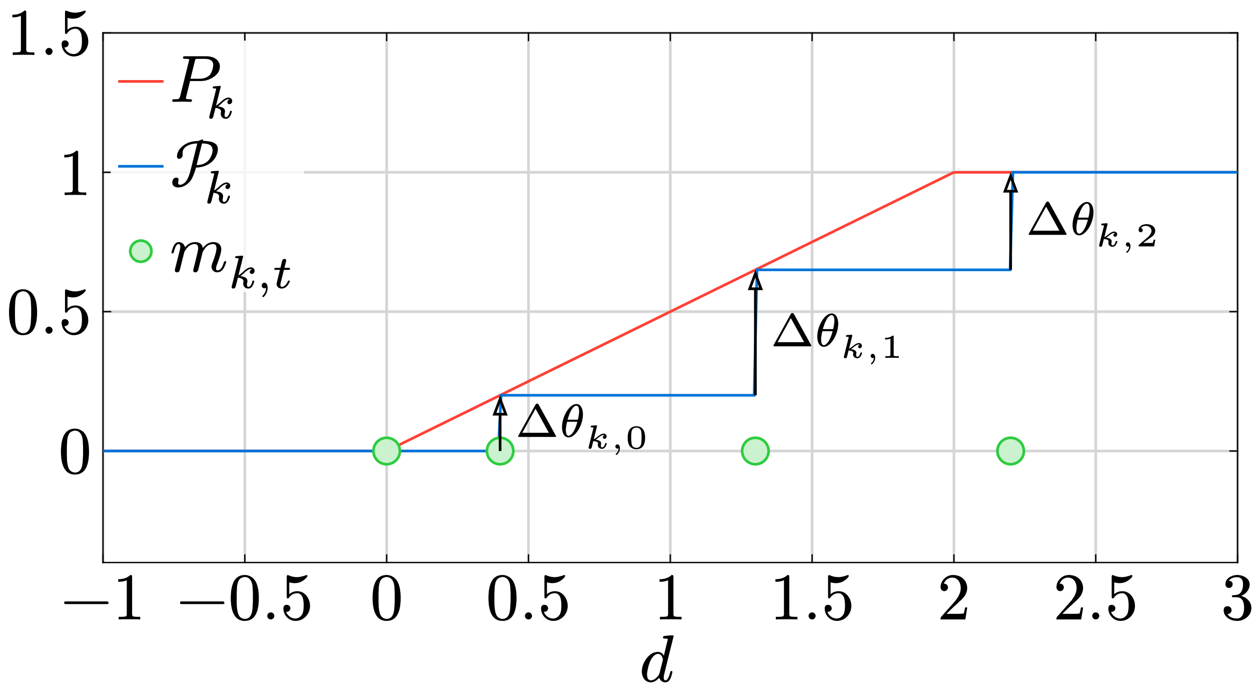

Figure 8: Layer cake preference function and preference increments.

The intra-criterion information is discretized as $\Delta\theta_{k,t} \geq 0$.

Computation of the NFS

Let the number of wins (resp. losses) of $a \in A$ at level $m_{k,t}$:

$$\begin{aligned} W_{a,k,t} &= \#\{b \in A \mid d_k(a,b) > m_{k,t}\}, \\ L_{a,k,t} &= \#\{b \in A \mid d_k(b,a) > m_{k,t}\}, \end{aligned} \tag{25}$$

and $s_{a,k,t} = W_{a,k,t} - L_{a,k,t}$ the win-loss balance at level $m_{k,t}$.

Theorem 1 (Linear net flow score) Using the above notations,

$$\phi(a) = \frac{1}{n-1} \sum_{k=1}^{q} \sum_{t=1}^{T_k} s_{a,k,t} \cdot w_k \Delta\theta_{k,t} \tag{26}$$

Some applications

Linear problems

Let

- $T = \sum_k T_k$ (= number of parameters),
- $V_a \in \mathbb{R}^T$ a stacking of $s_{a,k,t}$,
- $\beta_{k,t} = w_k \Delta\theta_{k,t}$, stacked into $\beta \in \mathbb{R}^T$.

Then

$$\phi(a) = \langle V_a, \beta \rangle \tag{27}$$

Feasibility problem

There is a set $\{w_k, P_k\}_{k=1}^{q}$ such that $a \in A$ can be ranked first iff $\exists \beta \in \mathbb{R}^T$ such that

$$\langle V_a, \beta \rangle > \langle V_b, \beta \rangle \quad \forall b \in A - \{a\} \tag{28}$$

This is a linear program with no objective function.

Robust Ordinal Regression

Regression

Suppose the DM can rank alternatives from $A^R \subsetneq A$. To study compatible parameters, one can solve

$$\text{Find } \beta \in \mathbb{R}^T, \text{ s.t. } \langle V_a - V_b, \beta \rangle > 0 \quad \forall (a,b) \in A^R \text{ with } a \succsim b. \tag{29}$$

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Extreme analysis

For a given pair $(a,b) \in A^2$, compute the best and worst possible difference of net flow scores

$$\max_{\|\beta\|=1} \langle V_a - V_b, \beta \rangle, \text{ and } \min_{\|\beta\|=1} \langle V_a - V_b, \beta \rangle. \tag{30}$$

Note that

$$w_k = \sum_{t=1}^{T_k} \beta_{k,t} \tag{31}$$

- Maximize and minimize w_k (as a linear objective) ;
- Enforcing some (partial) ranking using previous constrains.

This gives the WSIs of Mareschal.

Given shape of preference functions

Usual preference function

$$P_k(d) = \begin{cases} 0 \text{ if } d \leq 0 \\ 1 \text{ otherwise} \end{cases} \Rightarrow \begin{cases} \beta_{k,1} \geq 0 \\ \beta_{k,t} = 0 \text{ for } t = 2, \ldots, T_k \end{cases} \tag{32}$$

Add these as constraints to a LP for any criterion k to model with a usual preference function.

Further questions and considerations

- Parameter space is enormous, how to (effectively) reduce it while keeping expressivity ?
- We combined intra and inter-criteria parameters.
- What happens when performance table changes ?
- Construction of consensus ?
  - Each DM starts a “standard” P2 decision process
  - Remove constraints on PF shape and use current formalism to find “closest match”.

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Sensitivity analysis is mandatory in applied MCDA !

Bibliography

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U-Shape of parameter q_k

$$P_k(d) = \begin{cases} 0 \text{ if } d \leq q_k \\ 1 \text{ if } d > q_k \end{cases} \Rightarrow \exists! \beta_{k,t} \neq 0 \qquad (35)$$

Introduce **binary** variables $y_{k,t} \in \{0, 1\}$ to select where the "jump" occurs and enforce

$$\begin{gathered} \sum_{t=1}^{T_k} y_{k,t} = 1, \quad \text{(one jump)} \\ 0 \leq \beta_{k,t} \leq M \cdot y_{k,t}, \quad \text{for } t = 1, ..., T_k \end{gathered} \qquad (36)$$

Where M is large w.r.t the problem variables.