

Additive preferences over sets

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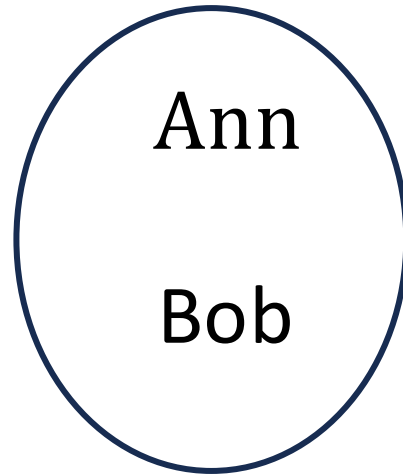
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Motivation

X



Agents



Ann



Bob



Fair division problem

X = set of indivisible goods.

$[n]$ = set of agents.

How to fairly allocate the objects to the agents?

A_1, A_2, \dots, A_n form a partition of X .

A_i = share of agent i (a subset of X).

Hypothesis 1: the preferences of each agent i can be represented by

$$v_i: 2^X \rightarrow \mathbb{R}_{\geq 0} \quad \text{with } v_i(\emptyset) = 0.$$

Hypothesis 2: each value function is additive, i.e.

$$\forall B \subseteq X, \quad v_i(B) = \sum_{x \in B} v_i(\{x\}).$$

Envy freeness (EF)

An allocation A is envy free if $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in [n]$.

Envy-freeness is not always possible.

An allocation A is envy free up to one good (EF1) if,

for all $i, j \in [n]$, there exists $x \in A_j$ such that $v_i(A_i) \geq v_i(A_j \setminus \{x\})$.

EF1 is always possible.

Pareto optimality

An allocation A is Pareto optimal if no other allocation A' can make some players strictly better off without making any player strictly worse off.

Formally: for all A' , if $\exists i \in [n]: v_i(A'_i) > v_i(A_i)$,
then $\exists j \in [n]: v_j(A'_j) < v_j(A_j)$.

Pareto optimality is always achievable.

Is it compatible with EF1?

Maximum Nash Welfare

Theorem

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang (2019), ACM Trans. Econ. Comput.]

If v_i is additive for all agents, then every allocation A maximizing $\prod_{i \in [n]} v_i(A_i)$ is EF1 and Pareto.

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Additive preferences

What does it mean that v_i is additive?

The function v_i is not observable.

Choices are observable.

E.g., when presented with $\{x, y, z\} \subset X$, the agent chooses x .

Under some conditions, choices can be rationalized by a preference relation \succsim .

Some preference relations admit an additive representation v :

for all finite non-empty subsets A, B of X ,

$$A \succsim B \iff v(A) \geq v(B)$$

$$v(A) = \sum_{x \in A} v(\{x\}).$$

When does an additive representation exist?

With X connected, when does an additive and continuous representation exist?

Axioms and result

\mathcal{X} = set of all finite subsets of X .

For all, $A, B, C \in \mathcal{X}$,

completeness: $A \succcurlyeq B$ or $B \succcurlyeq A$;

transitivity: $A \succcurlyeq B$ and $B \succcurlyeq C \implies A \succcurlyeq C$;

} Weak order

responsiveness: if $x \notin A \cup B$, then $A \succcurlyeq B \iff A \cup \{x\} \succcurlyeq B \cup \{x\}$;

continuity: the sets $\{x \in X: A \cup \{x\} \succcurlyeq B\}$ and $\{x \in X: A \cup \{x\} \preccurlyeq B\}$ are closed in the interval topology on X induced by the restriction of \succcurlyeq to singletons.

Theorem.

Suppose X is connected in the interval topology induced by the restriction of \succcurlyeq to singletons.

Assume also two (non-necessary) richness conditions.

The preference relation \succcurlyeq has a continuous additive representation iff the above axioms hold.

The representation is unique up to a linear transformation.

Richness conditions

For all, $A, B \in \mathcal{X}$,

Richness 1: $\exists C \in \mathcal{X} : A \sim C$ and $B \cap C = \emptyset$;

Richness 2: $\exists C \in \mathcal{X} : A \sim C$ and $\#C = \#A + 1$.

Example satisfying Richness 1 and 2: $\mathbb{R}_{>0}$ with $\succcurlyeq = \geq$.

$\mathbb{R}_{\geq 0}$ with $\succcurlyeq = \geq$ violates both richness conditions.

\mathbb{R} with $\succcurlyeq = \geq$ satisfies both richness conditions.

Comments and open questions

- The proof relies on Krantz, Luce, Suppes, and Tversky (1971) Foundations of Measurement
- Main behavioural axiom is Responsiveness. Well-known in the fair division literature.
- Are the axioms empirically valid?
- Can we weaken the richness conditions?