# Evaluation-based and ranking-based deepest voting

PEPR Condorcet- ANR-24-EXMA-0001

J.-B. Aubin, INSA-Lyon, ICJ
A. Rolland, Univ. Lyon 2, ERIC
I. Gavra, Univ. Rennes 2, IRMAR
I. Gannaz, UGA, G-SCOP
A. Kouassi, UGA, G-SCOP

**DA2PL**

April 16, 2026 – Bruxelles

RÉPUBLIQUE FRANÇAISE
Liberté
Égalité
Fraternité

FRANCE 2030 PROGRAMME DE RECHERCHE
MATHÉMATIQUES EN INTERACTION

cnrs

anr

# CONDORCET project

## CONsolidating DemOcRacy : Choosing an Electoral method Transparently

Comparing voting rules

- through axiomatic approach
- through simulations
- through experimentations in real life
- with large dissemination

A project mixing maths, stats, computer science, political science, experimental economy, ...

# Today's framework

- axiomatic approach
- $n$ voters, $m$ candidates
- $\mathcal{E} = [0, 1]$ or $\mathcal{E} = \{1, \dots, K\}$ or $\mathcal{E} = \mathfrak{S}_m$
- Social Choice Function

$$f \; : \; \mathcal{E}^n \mapsto W_{inner} \subseteq C_{andidates}$$

# Intuition

- **Kemeny-Young SFC** $(\mathcal{E} = \mathfrak{S}_m)$ : the winner is the best candidate in the median order.
- **Def** : the median order minimizes the sum of the Kendall tau distances to the voters' preference order.
- **What if**...
  - $\mathcal{E} = [0, 1]$ or $X = \{1, \dots, K\}$ or $\mathcal{E} = \mathfrak{S}_m$ ?
  - distance is not Kendall tau ?
  - sum is $p-$sum ?

# Deepest voting

- Data :
  - $\{\Phi(c,v) \in \mathcal{E}^{m\times n}; c \in \{c_1, \dots, c_m,\}; v \in \{v_1, \dots, v_n\}\}$
  - $D$ a depth function on $\mathcal{E}^m$
- Deepest set :

$$\mathcal{E}_D^* := \{x \in \mathcal{E}^m : D(x, \Phi) = \sup(D(., \Phi))\}$$

- Winner(s) :

$$c_D := \underset{1\leq c\leq m}{\operatorname{argmax}}\{x_{D,c}^*,\ x_D^* = \left(x_{D,1}^*, \dots, x_{D,m}^*\right) \in \mathcal{E}_D^*\}$$

# Depth functions

Let the mapping $D : \mathbb{R}^d \times \mathcal{F} \to \mathbb{R}$ be bounded and nonnegative. $D(\cdot;\cdot)$ is called a **statistical depth function** if it satisfies properties :

1. Let $X = (X_1, \ldots, X_d)$ be a random vector in $\mathbb{R}^d$, $x \in \mathbb{R}^d$, and $\sigma$ a permutation on $\{1, \ldots, d\}$. Then $D(x_\sigma, F_{X_\sigma}) = D(x; F_X)$.
2. For all $a \in \mathbb{R}$, $b \in \mathbb{R}^d$, for any random vector $X \in \mathbb{R}^d$, $\text{argmax}_{x\in\mathbb{R}^d} D(a\,x + b; F_{a\,X+b}) = \text{argmax}_{x\in\mathbb{R}^d} D(x; F_X)$.
3. For a distribution $F \in \mathcal{F}$ having a uniquely defined "center" $\theta$ (*e.g.* the point of "symmetry"), $D(\theta; F) = \sup_{x\in\mathbb{R}^d} D(x; F)$.
4. For any $F \in \mathcal{F}$, $D(\cdot; F)$ is quasi-concave.
5. $D(x; F) \to 0$ as $\|x\| \to \infty$ for each $F \in \mathcal{F}$.
6. Let $F \in \mathcal{F}$ be a distribution on $\mathbb{R}^d$ with marginal distributions $F_1, \ldots, F_d$. Suppose that for $i \in \{1, \ldots, d\}$, $F_i$ has support containing a unique point $\{\alpha\}$. Then for all $x^* \in \text{argsup}_{x\in\mathbb{R}^d} D(x; F)$, the $i^{\text{th}}$ coordinate of $x^*$ is $x_i^* = \alpha$.

# $L_q$ Deepest voting on evaluations [Aubin & al. 2022]

$L^q$ depth :

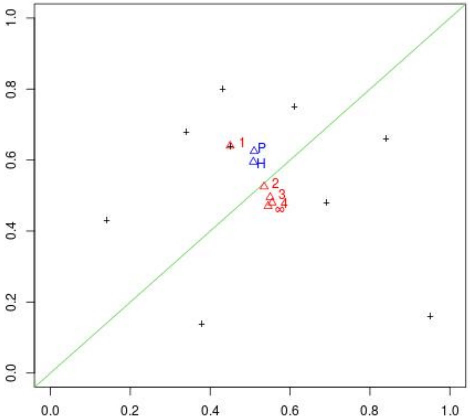
$$L^q D(x, \Phi) := \frac{1}{1 + \frac{1}{n} \sum_{v=1}^{n} \sum_{c=1}^{m} |\Phi(c, v) - x_c|^q}$$

| **Depth** | **Evaluations on** $\mathcal{E} = [0; 1]$ (continuous) |
|---|---|
| $L^1$ | Vote to the highest median (majority judgment) |
| $L^2$ | Vote to the highest mean (range voting) |
| $L^4$ | interesting new voting rule ! |

| **Depth** | **Evaluations on** $\mathcal{E} = \{0; 1\}$ (binary) |
|---|---|
| $L^p$, $p \geq 1$ | Approval voting |

| **Depth** | **Votes on** $\mathcal{E} = \mathfrak{S}_m$, depth on $[0; m]$ |
|---|---|
| $L^1$ | Bucklin's voting |
| $L^2$ | Borda count |

# Deepest voting [Aubin & al. 2022]




Each cross corresponds to a voter with grades on $c_1$ and $c_2$. Deepest points for the $L^p$ depths, for $p \in \{1, 2, 3, 4, \infty\}$, are displayed with labels $p$. Deepest points for halfspace depth and projection depth are displayed with respective labels $H$ and $P$.

# Deepest voting properties [Aubin & al. 2022]

Some properties about deepest voting rules.

- If $D$ a depth function satisfies P1 and P5, the deepest voting procedure $G_D$ associated to $D$ satisfies the properties Neutrality, Universality, and Unanimity.
- The $L^q$ depths family satisfies monotonicity
- Depth functions with a component-wise definition lead to voting processes satisfying IIA.
- For all $q \geq 1$, the $L^q$ deepest voting suffers from the Condorcet winner and the Condorcet loser paradoxes.
- $L^q$ deepest voting suffers from reinforcement and no-show paradoxes, for $q \in [1, \infty] \setminus \{2\}$. These properties do not hold for $L^2$ deepest voting.

# Depth function on ranking [Goibert & al. 2022]

- $\mathfrak{S}_m$ : set of permutations on $\{1, \ldots, m\}$
- $d()$ a distance on $\mathfrak{S}_m$ and $p \in \mathbb{R}$, $p \geq 1$ a given parameter
- $S$ a random variable of law $\Pi$

The depth function associated to $(d, p)$ is defined
$\forall \sigma \in \mathfrak{S}_m$, $\forall \Pi \in \mathcal{P}_m$,

$$D(\sigma, \Pi) = \|d^p\|_\infty - \mathbb{E}_{S \sim \Pi}[d^p(\sigma, S)]$$

with $\|d^p\|_\infty = \max_{\sigma, \tau \in \mathfrak{S}_m} d^p(\sigma, \tau)$

# $p$-Fréchet mean

- $(\mathfrak{S}_m, d)$ a metric space endowed with a probability measure $\Pi$
- for $p \geq 1$, the $p$-Fréchet mean on $(\mathfrak{S}_m, d)$ with respect to $\Pi$ is defined as

$$\operatorname*{argmin}_{\sigma \in \mathfrak{S}_m} U_{d,\Pi,p}(\sigma) = \operatorname*{argmin}_{\sigma \in \mathfrak{S}_m} \mathbb{E}_{S \sim \Pi} \left[ d^p(\sigma, S) \right]$$

# Deepest voting rule definition [Aubin & al. 2026]

- $\Phi_{m,n}$ a voting situation on $\mathfrak{S}_m^n$
- $\Pi$ the associated empirical probability measure on $\mathfrak{S}_m$,
- $d()$ a distance on $\mathfrak{S}_m$ (or pseudo-distance)
- $U_{d,\Pi,p}(\sigma) = \frac{1}{n}\sum_{j=1}^{n} d^p(\sigma, \Phi(.,j))$
- Deepest set of permutations :

$$\mathfrak{S}^*_{p,d,\Pi} := \{ \underset{\sigma\in\mathfrak{S}_m}{\text{argmax}}\, D(\sigma,\Pi)\} = \{ \underset{\sigma\in\mathfrak{S}_m}{\text{argmin}}\, U_{d,\Pi,p}(\sigma)\}$$

- Winner(s) :

$$i_D := \{i \in \{1,\dots,m\} \mid \sigma^*(i) = 1, \sigma^* \in \mathfrak{S}^*_{p,d,\Pi}\}$$

# Deepest voting rules [Aubin & al. 2026]

| Distance | Weights | $p$ | Voting rule |
|---|---|---|---|
| Kendall | none | 1 | Kemeny |
| Spearman $\rho$ | none | 2 | Borda |
| Hamming | $W(1, (1))$ | 1 | Plurality |
| Hamming | $W(-1, (m))$ | 1 | Antiplurality |

# Deepest voting on $[1, m]$ or $\mathfrak{S}_m$ ?

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $c_1$ | 1 | 1 | 4 | 4 | 3 |
| $c_2$ | 2 | 2 | 2 | 2 | 2 |
| $c_3$ | 3 | 3 | 3 | 3 | 1 |
| $c_4$ | 4 | 4 | 1 | 1 | 4 |

- median rankings : (3,2,3,4) ; Bucklin's winner : $c_2$
- optimum of the $p$-Fréchet mean ($p = 1, 2$) with the Spearman footrule distance : $(1, 2, 3, 4)$ ; winner : $c_1$

# Deepest voting rules properties [Aubin & al. 2026]

1. For $p = 1$, Kendall-based 1-Fréchet mean (consensus ranking) satisfies the Condorcet-winner property.
2. For $p > 1$, Kendall-based $p$-Fréchet mean does not satisfy the Condorcet-winner property.
3. For all $p \geq 1$, Cayley and Hamming based voting rules do not satisfy the Condorcet-winner property.
4. For $p = 1$ and all $q \geq 1$, $q$-Minkowski-Hölder based voting rules do not satisfy the Condorcet-winner property.

# Perspectives

(PhD. work in progress)

- Define new ranking-based voting rules based on classical depth functions.
- Study properties of voting rules using associated depth function.